

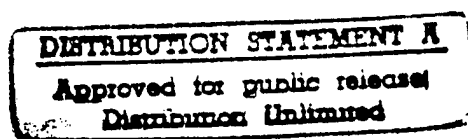
# **Frames and Orthonormal Bases for Variable Windowed Fourier Transforms**

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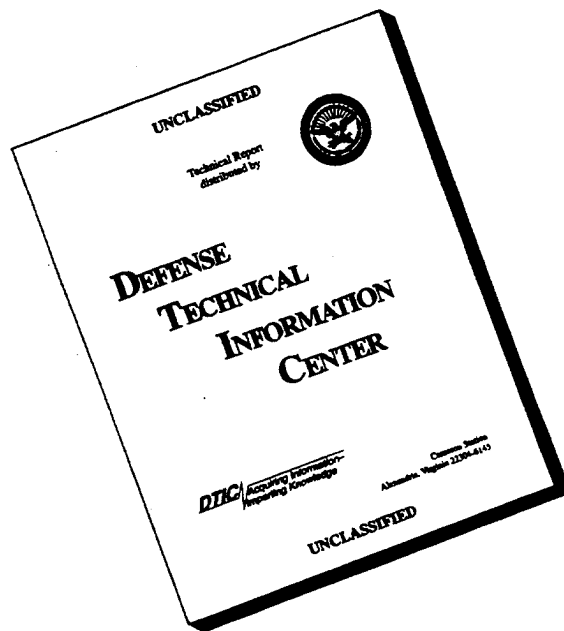
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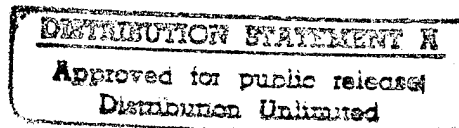
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# FRAMES AND ORTHONORMAL BASES FOR VARIABLE WINDOWED FOURIER TRANSFORMS

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## ABSTRACT

We generalize the windowed Fourier transform to the variable-windowed Fourier transform. This generalization brings the Gabor transform and the wavelet transform under the same framework. Using frame theory we characterize frames and orthonormal bases for the variable windowed Fourier series (VWFS). These characterizations are formulated explicitly in terms of window functions. Therefore they can serve as guidelines for designing windows for the VWFS. We introduce the notion of "complete orthogonal support" and, with the help of this notion, we construct a class of orthonormal VWFS bases for  $L^2(\mathbb{R}^+)$ .

## 1. INTRODUCTION

The Gabor transform [5] (or the windowed Fourier transform) is a widely used tool in signal processing. This transform uses a single window for the purpose of Fourier transforming a signal locally. This process is repeated while shifting the window through the real line. This single window shifting and modulation mechanism of the Gabor transform produces some undesirable effects [1, 6, 2].

In this paper we generalize the Gabor transform (or the windowed Fourier transform) to the variable windowed Fourier transform (VWFT). In this generalization, we relax the constraint of a single shifting window by allowing a set of windows. The implications of such a generalization are manifold. First, the control over windows is increased so that it is possible to design windows to get a tight frame or even an orthonormal basis. Secondly, it is possible to employ a parametric family of windows with useful properties [9]. Third, the VWFS can be made equivalent to a wavelet transform in the frequency-domain.

The VWFT comes in four varieties, just as the Fourier transform does. Adopting the convention of Fourier analysis, we call them the continuous VWFT or simply VWFT, the VWFS, the VWDTFT, and the VWDFFT [7]. In this paper we will only deal with the VWFS. We derive the condition for the existence of VWFS frames under a mild assumption. When a frame exists, we give formulae for dual frames, tight frames, and orthonormal bases. Of course we also give the inverse transform. We also present several examples of frames and orthonormal bases for the VWFS.

In constructing orthonormal bases for the VWFS, we have found the concept of "complete orthogonal support" to be useful. C. Wei and D. Cochran bring up the concept

of "orthogonal support" and use this concept to construct Bandlimited Orthonormal Wavelet sets [11]. An orthogonal support is basically a support set which never overlaps with its diadic dilations. We need a support which is not only orthogonal but also "complete". We will formalize the notion of "complete" in a later Section. In [11] a construction of an  $n$ -band orthogonal support is given, which is in fact complete. We show, with some small constraint, that a complete  $n$ -band orthogonal support leads to an orthonormal basis for the VWFS.

The remainder of this paper is organized as follows: In Section 2 a brief review of frames is presented. In Section 3 we introduce orthogonal support, complete orthogonal support, and a construction of a complete orthogonal support. Then the VWFT and VWFS are defined in Section 4. Characterizations of VWFS frames, dual frames, tight frames, and orthonormal bases are given in Section 5. Some examples of frames and orthonormal bases for VWFS are presented in Section 6.

## 2. A BRIEF REVIEW OF FRAMES

In this section, we briefly review some generalities about frames. Theorems are stated without proofs. For more detailed treatments of frames, refer to [3, 12].

A frame is a set of dependent or independent vectors which can be used to write an explicit expansion for every vector in the space. An orthonormal basis is just a special frame. The definition of frames is given as follows.

**Definition 1 (Frames)** A set of vectors  $(\phi_k)_{k \in \mathbb{K}}$  in a Hilbert space  $\mathcal{H}$  is called a frame if there exist constants  $A > 0, B < \infty$  so that for all  $f$  in  $\mathcal{H}$

$$A\|f\|^2 \leq \sum_{k \in \mathbb{K}} |\langle f, \phi_k \rangle|^2 \leq B\|f\|^2 \quad (1)$$

$A$  and  $B$  are called frame bounds.  $\square$

If the two frame bounds are equal, namely  $A = B$ , then the frame is called a tight frame.

**Definition 2 (Frame Operators)** The frame operator  $S : \mathcal{H} \rightarrow \mathcal{H}$  is defined as

$$Sf = \sum_{k \in \mathbb{K}} \langle f, \phi_k \rangle \phi_k \quad (2)$$

$\square$

**Theorem 1 (Dual Frame)** Let  $(\phi_k)_{k \in \mathbb{K}}$  be a frame in  $\mathcal{H}$  with lower bound  $A$  and upper bound  $B$  and let  $\tilde{\phi}_k = S^{-1}\phi_k$ . Then  $(\tilde{\phi}_k)_{k \in \mathbb{K}}$  is also a frame in  $\mathcal{H}$  with lower bound  $B^{-1}$  and upper bound  $A^{-1}$ .  $\square$

The family  $(\tilde{\phi}_k)_{k \in \mathbb{K}}$  is called the dual frame of  $(\phi_k)_{k \in \mathbb{K}}$ .

**Theorem 2 (Frame Expansions)** Let  $(\phi_k)_{k \in \mathbb{K}}$  be a frame in  $\mathcal{H}$  with  $(\tilde{\phi}_k)_{k \in \mathbb{K}}$  its dual frame. Then for all  $f$  in  $\mathcal{H}$ , the following is true:

$$\sum_{k \in \mathbb{K}} \langle f, \phi_k \rangle \tilde{\phi}_k = f = \sum_{k \in \mathbb{K}} \langle f, \tilde{\phi}_k \rangle \phi_k. \quad (3)$$

$\square$

**Theorem 3 (Orthonormal Basis)** If  $(\phi_k)_{k \in \mathbb{K}}$  is a tight frame, with frame bound  $A = 1$ , and if  $\|\phi_k\| = 1$  for all  $k \in \mathbb{K}$ , then  $(\phi_k)_{k \in \mathbb{K}}$  constitute an orthonormal basis.  $\square$

### 3. ORTHOGONAL SUPPORT

A support is a union of disjoint intervals. A support is orthogonal if its diadic dilations never overlap. We borrow the following definition from [11].

**Definition 3 (Orthogonal Support)** Let a support  $S_0$  be the union of  $N$  disjoint intervals

$$S_0 = [s_0, s_1] \cup [s_2, s_3] \cup \dots \cup [s_{2N-2}, s_{2N-1}] \quad (4)$$

and let  $S_m$  be the  $m$ -th dilation of  $S_0$ :

$$S_m = [s_0 2^{-m}, s_1 2^{-m}] \cup [s_2 2^{-m}, s_3 2^{-m}] \cup \dots \cup [s_{2N-2} 2^{-m}, s_{2N-1} 2^{-m}] \quad (5)$$

If the measure of  $S_m \cap S_{m'}$  is zero for all integers  $m \neq m'$ ,  $S_0$  is an orthogonal support.  $\square$

We now formalize the notion of "complete". A support is complete if the union of all its diadic dilations is the support of the underlying Hilbert space.

**Definition 4 (Complete Orthogonal Support)** In some Hilbert space  $\mathcal{H}$ ,  $S_0$  is called a complete orthogonal support if  $S_0$  is an orthogonal support and  $\bigcup_{m \in \mathbb{Z}} S_m = \text{supp}(\mathcal{H})$ .  $\square$

A construction of an orthogonal support is given in [11]. An orthogonal support thus constructed is indeed orthogonal and, moreover, complete for  $\mathcal{H}$  where  $\text{Supp}(\mathcal{H}) = \mathbb{R}^+$ . For convenience we modify the result of Wei and Cochran [11] into the following Theorem:

**Theorem 4 (Orthogonal Support)** Let  $\text{Supp}(\mathcal{H}) = \mathbb{R}^+$ . Consider real numbers  $0 < c_0 < c_1 < \dots < c_N = 2c_0$ . Then

$$[c_0, 2c_0] = [c_0, c_1] \cup [c_1, c_2] \cup \dots \cup [c_{N-1}, 2c_0]$$

Let  $q_1, q_2, \dots, q_N$  be distinct integers and form  $S_0$ :

$$S_0 = [c_0 2^{q_1}, c_1 2^{q_1}] \cup [c_1 2^{q_2}, c_2 2^{q_2}] \cup \dots \cup [c_{N-1} 2^{q_N}, 2c_0 2^{q_N}] \quad (6)$$

$$= [s_0, s_1] \cup [s_2, s_3] \cup \dots \cup [s_{2N-2}, s_{2N-1}] \quad (7)$$

Then  $S_0$  is an  $N$ -band orthogonal support for  $\mathcal{H}$ .  $\square$

### 4. VWFT AND VWFS

**Definition 5 (VWFT)** In some Hilbert space  $\mathcal{H} \subseteq L^2(\mathbb{R})$  the VWFT of  $f(t)$  is

$$(\mathcal{F}_v f)(\omega, n) = \langle f, \phi_{\omega n} \rangle \quad (8)$$

$$= \langle f(t), p_n(t) e^{j\omega t} \rangle \quad (9)$$

where  $\{p_n(t)\}$  is a set of window functions,  $n \in$  some index set which can be continuous or discrete.  $\square$

Examples are: (1) For  $\mathcal{H} = L^2(\mathbb{R})$ , if  $p_n(t) = g(t-n)$  and  $n \in \mathbb{R}$  then the VWFT specializes to the continuous Gabor transform. (2) For  $\mathcal{H} = L^2(\mathbb{R}^+)$ , if  $p_n(t) = \gamma^{1/2}(t; n, \alpha)$ , where  $\gamma(t; n, \alpha)$  is the gamma probability density function of degree  $n$  with parameter  $\alpha$ , then the VWFT specializes to the continuous Gamma transform [9]. (3) For  $\mathcal{H} = L^2(\mathbb{R})$ , if  $p_n(t) = |n|^{1/2} \psi(nt)$ ,  $\psi(0) = 0$  and  $n \in \mathbb{R}$ , then the VWFT is equivalent to a continuous wavelet transform in the frequency-domain.

To proceed from a VWFT to a VWFS one needs to sample frequency. Since we now allow a variable set of window functions  $\{p_n(t)\}$ , we expect this frequency sampling to be dependent on  $p_n(t)$ .

**Definition 6 (VWFS)** In some Hilbert space  $\mathcal{H}$  the VWFS of  $f(t)$  is

$$(\mathcal{F}_v f)(m, n) = \langle f, \phi_{mn} \rangle \quad (10)$$

$$= \langle f(t), p_n(t) e^{jm\omega_n t} \rangle \quad (11)$$

where  $\{p_n(t)\}$  is a set of window functions,  $m \in \mathbb{Z}$ ,  $n \in$  some discrete set, and  $\omega_n > 0$ .  $\square$

We call  $\omega_n$  the fundamental frequency associated with the window  $p_n(t)$  and  $m\omega_n$  the  $m$ -th harmonic of the fundamental frequency. Two especially interesting ways of determining fundamental frequency are: (1)  $\omega_n = \omega_0$  (homogeneous fundamental frequency) and (2)  $\omega_n = \omega_0 \sigma_0^n$ ,  $\sigma_0 > 1$  (exponential fundamental frequency). The Gabor transform is a homogeneous VWFS and the wavelet transform in the frequency-domain is an exponential VWFS. This brings the Gabor transform and the wavelet transform under the same framework.

### 5. FRAMES AND ORTHONORMAL BASES

In this section we characterize frames, dual frames, tight frames, and orthonormal bases for the VWFS assuming the frequency sampling is exponential. The homogeneous VWFS has been treated separately in [10]. We shall assume  $p_n(t) = \sigma_0^{n/2} \psi(\sigma_0^n t)$  and therefore  $\phi_{mn} = \sigma_0^{n/2} \psi(\sigma_0^n t) e^{jm\omega_0 \sigma_0^n t}$ . The Fourier transform of  $\phi_{mn}$  is the  $mn$ -th wavelet in the frequency domain.

In order to make the characterization simple and useful, we need the following assumption:

$$\text{Supp}(\psi(t)) \leq \frac{2\pi}{\omega_0} \quad (12)$$

**Theorem 5 (Frames)** With assumption (12), if there exist constants  $0 < C \leq D < \infty$  and  $\sigma_0 > 1$  such that

$$C \leq \sum_{n \in \mathbb{Z}} |\psi(\sigma_0^n t)|^2 \leq D \text{ for } t \in [1, \sigma_0] \quad (13)$$

then  $\{\phi_{mn}\}$  constitute a frame with frame bounds  $(2\pi/\omega_0)C$  and  $(2\pi/\omega_0)D$ .  $\square$

**Proof:**

$$\begin{aligned} & \sum_{mn} |(\mathcal{F}_v f)(m, n)|^2 \\ &= \sum_{mn} \left| \int_{-\infty}^{\infty} dt f(t) \overline{p_n(t)} e^{-jm\omega_n t} \right|^2 \\ &= \sum_{mn} \left| \int_0^{\frac{2\pi}{\omega_n}} dt e^{-jm\omega_n t} \sum_{l \in \mathbb{Z}} f(t + l\frac{2\pi}{\omega_n}) \overline{p_n(t + l\frac{2\pi}{\omega_n})} \right|^2 \\ &= \sum_n \frac{2\pi}{\omega_n} \int_0^{\frac{2\pi}{\omega_n}} dt \left| \sum_{l \in \mathbb{Z}} f(t + l\frac{2\pi}{\omega_n}) \overline{p_n(t + l\frac{2\pi}{\omega_n})} \right|^2 \\ &= \sum_n \frac{2\pi}{\omega_n} \sum_{l \in \mathbb{Z}} \int_{-\infty}^{\infty} dt f(t) \overline{f(t + l\frac{2\pi}{\omega_n})} \overline{p_n(t)} p_n(t + l\frac{2\pi}{\omega_n}) \end{aligned}$$

With  $p_n(t) = \sigma_0^{n/2} \psi(\sigma_0^n t)$ ,  $\omega_n = \omega_0 \sigma_0^n$ , and assumption (12), the RHS of last equation becomes

$$\frac{2\pi}{\omega_0} \int_{-\infty}^{\infty} dt |f(t)|^2 \sum_{n \in \mathbb{Z}} |\psi(\sigma_0^n t)|^2$$

But  $C \leq \sum_n |\psi(\sigma_0^n t)|^2 \leq D$  for  $t \in [1, \sigma_0]$ , hence

$$(2\pi/\omega_0)C \|f\|^2 \leq \sum_{mn} |\langle f, \phi_{mn} \rangle|^2 \leq (2\pi/\omega_0)D \|f\|^2$$

**Theorem 6 (Frame Operator)** Suppose eq(13) is true, and assumption (12) holds. Then the frame operator  $S$  of an exponential VWFS is

$$S = \frac{2\pi}{\omega_0} \sum_{n \in \mathbb{Z}} |\psi(\sigma_0^n t)|^2 I \quad (14)$$

**Proof:**

$$\begin{aligned} & S f(t) \\ &= \sum_{mn} \langle f, \phi_{mn} \rangle \phi_{mn} \\ &= \sum_{mn} \left[ \int_{-\infty}^{\infty} ds f(s) \overline{p_n(s)} e^{-jm\omega_n s} \right] p_n(t) e^{jm\omega_n t} \\ &= \int_{-\infty}^{\infty} ds f(s) \sum_n \overline{p_n(s)} p_n(t) \sum_m e^{jm\omega_n(t-s)} \\ &= \int_{-\infty}^{\infty} ds f(s) \sum_n \overline{p_n(s)} p_n(t) \sum_r \frac{2\pi}{\omega_n} \delta(t-s-r\frac{2\pi}{\omega_n}) \\ &= \sum_n \frac{2\pi}{\omega_n} \sum_r f(t-r\frac{2\pi}{\omega_n}) \overline{p_n(t-r\frac{2\pi}{\omega_n})} p_n(t) \end{aligned}$$

Since  $p_n(t) = \sigma_0^{n/2} \psi(\sigma_0^n t)$ ,  $\omega_n = \omega_0 \sigma_0^n$ , and assumption (12) holds, the RHS of last equation becomes

$$\frac{2\pi}{\omega_0} \sum_{n \in \mathbb{Z}} |\psi(\sigma_0^n t)|^2 f(t)$$

Therefore  $S = \frac{2\pi}{\omega_0} \sum_{n \in \mathbb{Z}} |\psi(\sigma_0^n t)|^2 I$ .  $\blacksquare$

The dual frame  $\widetilde{\phi_{mn}}$  of the VWFS is

$$\widetilde{\phi_{mn}}(t) = S^{-1} \phi_{mn}(t) \quad (15)$$

**Theorem 7 (Tight Frames)** Suppose eq(13) is true, and assumption (12) holds. Then  $\phi_{mn}$  constitute a tight frame with frame bound  $\frac{2\pi}{\omega_0}c$  if and only if

$$\sum_{n \in \mathbb{Z}} |\psi(\sigma_0^n t)|^2 = c \text{ for } t \in [1, \sigma_0] \quad (16)$$

Furthermore if  $\|\psi\| = 1$  and  $\frac{2\pi}{\omega_0}c = 1$ ,  $\{\phi_{mn}\}$  constitute an orthonormal basis.  $\square$

**Proof:** This is true because of Theorem 6  $\blacksquare$

**Theorem 8 (Inverse VWFS)** Suppose eq(13) is true, and assumption (12) holds. Then the inverse VWFS is

$$f(t) = \frac{1}{\frac{2\pi}{\omega_0} \sum_{n \in \mathbb{Z}} |\psi(\sigma_0^n t)|^2} \sum_{mn} \langle f, \phi_{mn} \rangle \phi_{mn} \quad (17)$$

**Proof:** This is true because of Theorem 2.  $\blacksquare$

## 6. EXAMPLES

In this Section we present some examples of frames, tight frames, and orthonormal bases for the VWFS. All examples are for  $\mathcal{H} = L^2(\mathbb{R}^+)$ .

### 6.1. A Frame

Let  $\psi_k(t) = \gamma^{1/2}(t; k, 2)$  with  $\gamma(t; k, 2)$  the gamma pdf with degree  $k$  and parameter 2. Note that  $\|\psi(t)\| = 1$ . Let  $p_n(t) = \sigma_0^{n/2} \psi(\sigma_0^n t)$  for some  $\sigma_0 > 1$  and therefore  $\phi_{mn} = \sigma_0^{n/2} \psi(\sigma_0^n t) e^{jm\omega_0 \sigma_0^n t}$ . We have shown in [8] that  $\{\phi_{mn}\}$  constitute a VWFS frame which is nearly tight for a fairly wide range of  $k$  and  $\sigma_0$ . This VWFS resolves causal signals onto the complex frequency plane and has very good time-frequency resolution. Each frame vector  $\phi_{mn}$  can be rearranged as a complex exponential with "delay"  $k$  and "damping"  $\sigma_0$ . Therefore this VWFS works like infinite-dimensional modal analysis using complex exponential modes. Note that  $\psi_k(t)$  is not finite support. Thus the Theorems in Section 5. are not applicable here.

## 6.2. Non-overlapped ON bases

By Theorem 3, to design an orthonormal basis for the VWFS is to seek a tight frame with unit bound and unit norm windows. If a window has finite support, then by Theorem 7, (16) becomes a sufficient condition for an orthonormal basis. Note that (16) is essentially a requirement that the windows partition unity. The requirement of (16) is especially easy to achieve when the windows do not overlap.

### Theorem 9 (Non-overlapped ON bases)

Consider real numbers  $0 < c_0 < c_1 < \dots < c_N = 2c_0$ . Then

$$[c_0, 2c_0] = [c_0, c_1] \cup [c_1, c_2] \cup \dots \cup [c_{N-1}, 2c_0]$$

let  $d_k = c_k - c_{k-1}$  for  $1 \leq k \leq N$ . If  $d_k = 2^{l_k} c_0$  for some integer  $l_k$ , then  $\{\phi_{mn}^k\}_{k=1}^N$  constitute an orthonormal basis for  $L^2(\mathbf{R}^+)$  where

$$\phi_{mn}^k = 2^{n/2} \psi^k(2^n t) e^{j m \omega_0 2^n t} \text{ for } 1 \leq k \leq N$$

$$\omega_0 = \frac{2\pi}{c_0}$$

$$\psi^k(t) = \sqrt{\frac{1}{c_0}} \chi_{[c_{k-1} 2^{-l_k}, c_k 2^{-l_k}]} e^{j \theta_k(t)}$$

$\chi_{[a,b]}$  is the characteristic function on interval  $[a, b]$

$\theta_k(t)$  is an arbitrary finitely oscillatory phase function, taking only values 0 or  $\pi$ , that turns  $\psi$  into a sequence of polarity-switched square pulses.

□

**Proof:** let  $q_1, q_2, \dots, q_N$  be distinct integers and form  $S_0$

$$\begin{aligned} S_0 &= [c_0 2^{q_1}, c_1 2^{q_1}] \cup [c_1 2^{q_2}, c_2 2^{q_2}] \cup \dots \cup \\ &\quad [c_{N-1} 2^{q_N}, 2c_0 2^{q_N}] \\ &= [s_0, s_1] \cup [s_2, s_3] \cup \dots \cup [s_{2N-2}, s_{2N-1}] \\ &= S_0^1 \cup S_0^2 \cup \dots \cup S_0^N \end{aligned}$$

From Theorem 4

$$\begin{aligned} S_0 &\text{ is a complete orthogonal support for } L^2(\mathbf{R}^+) \\ \Rightarrow &\text{ so is } \{S_0^1, S_0^2, \dots, S_0^N\} \\ \Rightarrow &\text{ so is } \{S_0^1 2^{-(q_1+l_1)}, S_0^2 2^{-(q_2+l_2)}, \dots, S_0^N 2^{-(q_N+l_N)}\} \\ \Rightarrow &\text{ so is } \{[c_0 2^{-l_1}, c_1 2^{-l_1}], [c_1 2^{-l_2}, c_2 2^{-l_2}], \dots, \\ &\quad [c_{N-1} 2^{-l_N}, c_N 2^{-l_N}]\} \end{aligned}$$

$$\text{Let } \psi^k(t) = \sqrt{\frac{1}{c_0}} \chi_{[c_{k-1} 2^{-l_k}, c_k 2^{-l_k}]} e^{j \theta_k(t)}$$

Clearly,  $\text{supp}(\{\psi^k(2^n t)\}_{k=1}^N)$  tessellate  $\mathbf{R}^+$ . Furthermore  $\|\psi^k(t)\| = 1$  and  $|\psi^k(2^n t)|^2 = \frac{1}{c_0} = \frac{\omega_0}{2\pi}$ . Therefore  $\{\phi_{mn}^k\}_{k=1}^N$  constitute an orthonormal basis for  $L^2(\mathbf{R}^+)$ . ■

**Example 1** By Theorem 9  $\{\phi_{mn}\}$  is an orthonormal basis in  $L^2(\mathbf{R}^+)$  for  $N = 1, c_0 = 1, d_1 = 1, \omega_0 = 2\pi$ , and  $\psi(t) = \chi_{[1,2]}$ .

**Example 2** By Theorem 9  $\{\phi_{mn}^k\}_{k=1}^3$  is an orthonormal basis in  $L^2(\mathbf{R}^+)$  for  $N = 3, c_0 = 1, d_1 = 1/4, d_2 = 1/2, d_3 = 1/4, \omega_0 = 2\pi$ , and  $\psi^1(t) = \chi_{[4,5]}, \psi^2(t) = \chi_{[5/2, 7/2]}, \psi^3(t) = \chi_{[7,8]}$ .

## 6.3. Overlapped ON bases

Note that all orthonormal bases promised by Theorem 9 have square-wave shape. This follows from the requirement that windows do not overlap. Without this constraint it is possible to construct smooth orthonormal VWFS bases.

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